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Effects of thermal radiation in magnetohydrodynamic channel flow

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SUMMARY
An exact solution is obtained for the Hartmann problem of the viscous laminar flow of an electrically conducting liquid between parallel walls in the presence of a transverse magnetic field when effects of thermal radiation are significant and thermal conductivity may be neglected.

1. Introduction

A simple yet basic problem in magnetohydrodynamics consists of the analysis of the steady viscous laminar flow of an electrically conducting liquid between parallel plane walls in the presence of a transverse magnetic field and a streamwise pressure gradient. A study of this (commonly called) Hartmann flow leads to an understanding of the fundamentals of magnetohydrodynamic pump, generator and flow meter design, as well as laying a foundation for an investigation into a host of other devices incorporating viscous flow phenomena. In numerous applications however the liquid is likely to be sufficiently hot for effects of thermal radiation to be significant and this paper is thus concerned with an examination of radiative Hartmann flow.

2. The governing equations

We consider flow, parallel to the x-axis, down a long channel of great width in the z-direction between walls parallel to the xz–plane distance $2h$ apart, where $Oxyz$ comprise a set of orthogonal cartesian axes with origin midway between the walls. The bounds of the channel normal to the z-axis are taken to be electrodes whilst the walls normal to the y-axis are insulators. In order to study radiative effects in a simple configuration it is assumed that the electrodes are perfect conductors and the walls perfect insulators whilst the liquid electrical conductivity is finite. Furthermore it is supposed that the electrodes are set infinitely far apart. The problem then becomes one-dimensional and with the possible exception of the pressure all variables are functions of y alone.

With an external magnetic field $\mathbf{B}_0$ applied uniformly across the channel normal to the insulator walls the analysis of the equations of momentum, continuity and electromagnetism is classical, as presented for instance in the text by Boyd and Sanderson [1]. One obtains the following results:

The electric field has a single component, E_0, which is constant and directed parallel to the z-axis.

The electric current similarly has a single component, j, acting in the same direction.

The magnetic field has a component B_x induced along the length of the channel whilst the component parallel to the y-axis remains constant with magnitude B_0.

The pressure p takes the form

$$p = -p_0 x + p'$$

where p_0 is the pressure gradient down the channel. The transverse variation $p' = p'(y)$ is given by the result that the total "pressure" $p' + \frac{1}{2}B_x^2/\mu$ remains constant, where μ is the permeability.

Then if σ denotes the electrical conductivity of the liquid and $\mathbf{u}$ its velocity down the channel it follows that

$$\frac{d}{dy}\left(\mu \frac{du}{dy}\right) - \sigma B_0^2 u + (p_0 - \sigma E_0 B_0) = 0 \tag{1}$$

$$j = -\frac{1}{\mu}\frac{dB_x}{dy} = \sigma(E_0 + uB_0)\,. \tag{2}$$

It remains to consider the energy equation. For a thermally conducting viscous liquid the effects of radiative heat transfer give rise to a term involving the radiative flux, q, which in this problem has a single component parallel to the y-axis. For temperatures which are not extremely high the contributions from the radiative energy density and radiative pressure may be neglected (the contribution from the latter has already been neglected in the analysis of the momentum equation) and the equation becomes

$$\frac{d}{dy}\left(k \frac{dT}{dy}\right) + \eta\left(\frac{du}{dy}\right)^2 - \frac{dq}{dy} + \sigma(E_0 + uB_0)^2 = 0\,, \tag{3}$$

where k is the thermal conductivity, η the coefficient of viscosity and T is the temperature of the liquid.

The closure of the set of governing equations requires a statement of the equation of radiative transfer and its relationship to the radiative flux. It is well known that this gives rise to a coupled system of integro-differential equations, somewhat intractable to analysis. Consequently various forms of approximation have been employed to simplify the formulation and to generate a system of equations which are entirely differential. One such approximation is indeed termed the differential approximation and has been shown to be particularly appropriate for use with one dimensional problems, see for instance the survey by Vincenti and Kruger [2]. We take the form relevant to a fluid of general opacity, viz.

$$\frac{d^2q}{dy^2} - 3\alpha^2 q = \frac{d}{dy}(\tilde{\sigma}\alpha T^4)\,, \tag{4}$$

where α is the volumetric absorption coefficient averaged over all radiative frequencies, and $\tilde{\sigma}$ is Stefan's constant.

Finally it is necessary to specify the boundary conditions at the channel walls. Cess [3] has recently discussed the form of the radiative conditions appropriate to the differential approximation for the case of non-black walls. Thus if the lower wall has emissivity ε_1 and is maintained at temperature T_1 whilst the upper wall has emissivity ε_2 and temperature T_2 the radiative boundary conditions are

$$4\tilde{\sigma}T^4 - \left(\frac{4}{\varepsilon_2} - 2\right) q - \frac{1}{\alpha}\frac{dq}{dy} = 4\tilde{\sigma}T_2^4; \quad y = h : \tag{5}$$

$$4\tilde{\sigma}T^4 + \left(\frac{4}{\varepsilon_1} - 2\right) q - \frac{1}{\alpha}\frac{dq}{dy} = 4\tilde{\sigma}T_1^4; \quad y = -h\,. \tag{6}$$

The no-slip condition for the fluid at the walls is expressed as

$$u = 0\,, \quad y = \pm h\,. \tag{7}$$

If the fluid is thermally non-conducting, these comprise a complete set of boundary conditions. However in the case when thermal conductivity is non-zero they must be supplemented by the requirement that the temperatures of a wall and fluid adjacent are continuous so that, respectively,

$$T = T_1, T_2\ ; \quad y = h, -h \tag{8}$$

with corresponding adjustments to the conditions (5) and (6).

It will be observed that equations (1) and (2) separate from (3) and (4) so that for radiative Hartmann flow the profiles of the velocity, current and magnetic field are not influenced by radiation. Thus, solving these independently, we introduce

$$P = \left(\frac{p_0 h\mu}{B_0^2}\right)\left(\frac{1}{\mu h\sigma U_0}\right) = \frac{\text{Pressure Ratio}}{\text{Magnetic Reynolds No.}}$$

$$J = \left(\frac{j_0 h\mu}{B_0}\right)\left(\frac{1}{\mu h\sigma U_0}\right) = \frac{\text{Current Ratio}}{\text{Magnetic Reynolds No.}}$$

$$M = B_0 h\left(\frac{\sigma}{\mu}\right)^{\frac{1}{2}} = \text{Hartmann No.}$$

where U_0 is the mean velocity and j_0 the mean current density. Then it follows that

$$P = J + \frac{1}{M \coth M - 1},$$

and the electric field is given by

$$\boldsymbol{E} = \frac{E_0}{U_0 B_0} = J - 1\,.$$

The variation with position across the channel of the velocity, current density and induced magnetic field (taking as boundary condition for the last, symmetry about the central plane of the channel) may be written

$$\frac{u}{U_0} = \frac{M(\cosh M - \cosh M\zeta)}{M \cosh M - \sinh M},$$

$$\frac{j}{\sigma U_0 B_0} = J - 1 + \frac{u}{U_0},$$

$$\frac{B_x}{\mu\sigma h U_0 B_0} = \frac{\sinh M\zeta}{M \cosh M - \sinh M} - \zeta\left\{\frac{M \cosh M}{M \cosh M - \sinh M} + J - 1\right\},$$

where $\zeta = y/h$.

3. The solution with zero thermal conductivity

An exact solution of the problem may be obtained in the case of zero thermal conductivity and constant absorption coefficient. It is then useful to introduce the following notation

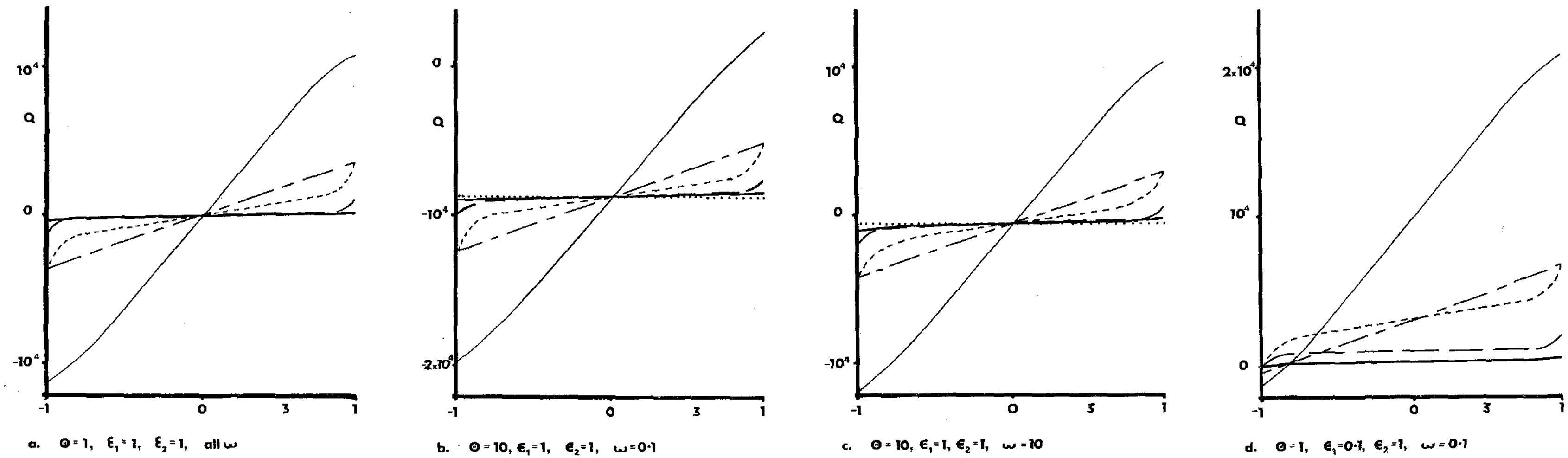
$$\Theta = \frac{T_2}{T_1} = \text{temperature ratio across the channel}\,,$$

$$\omega = \alpha h = \text{Bouguer No.}\,,$$

$$N = \left(\frac{\tilde{\sigma} T_1^4}{\rho_0 U_0^3}\right)\left(\frac{h\rho_0 U_0}{\eta}\right) = \frac{\text{Reynolds No.}}{\text{Boltzmann No.}},$$

$$\theta = \frac{T}{T_1} = \text{dimensionless temperature}\,,$$

$$Q = \frac{q}{\tilde{\sigma} T_1^4} = \frac{\text{radiative flux}}{\text{black wall emissive power}}\,.$$



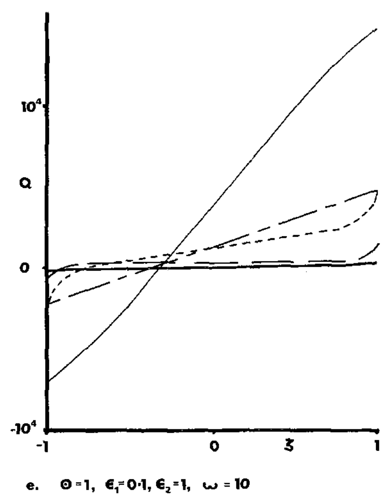


Figure 1. *Radiative flux.*

Cases a, d, e: $N = 0.01$ ordinate stated: ——— $M = 0$
$N = 1$ ordinate $\times 10^{-2}$: ——— $J = 1$
$N = 100$ ordinate $\times 10^{-4}$: — — — $J = 0.5$
- - - - - $J = -0.5$
— — — $J = 0$
} $M = 10$

Cases b, c: $N = 0.01$ as above
$N = 100$, $0 < M < 10$, all J ...

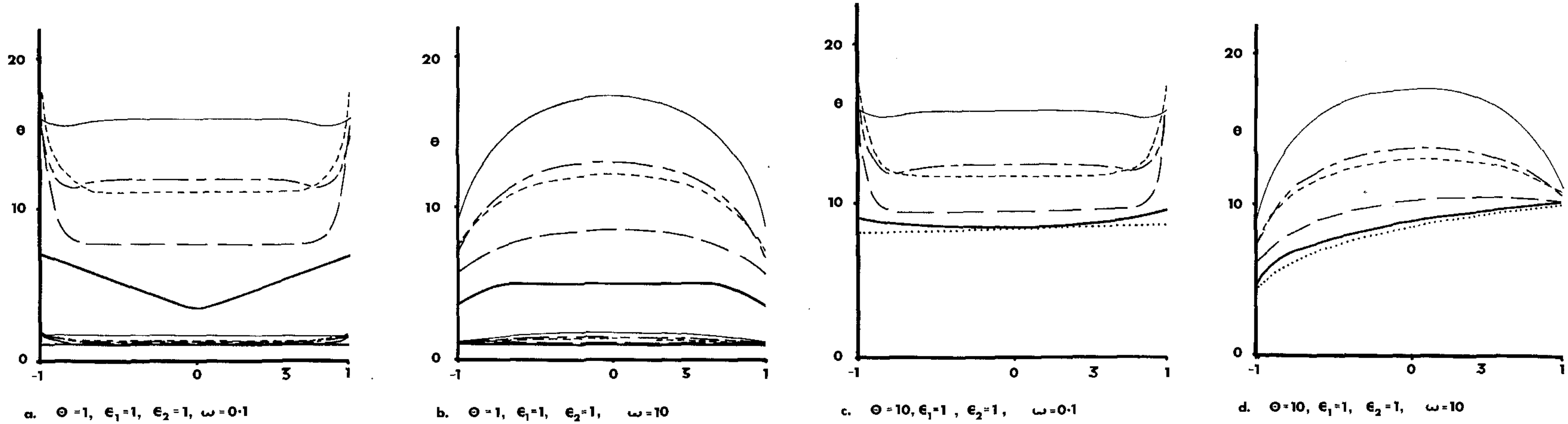
(a) $\Theta = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, all ω
$Q(-10^4, 0, 10^4)$, $\zeta(-1, 0, 1)$

(b) $\Theta = 10$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\omega = 0.1$
$Q(-2\times 10^4, -10^4, 0)$ $\zeta(-1, 0, 1)$

(c) $\Theta = 10$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\omega = 10$
$Q(-10^4, 0, 10^4)$, $\zeta(-1, 0, 1)$

(d) $\Theta = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$, $\omega = 0.1$
$Q(0, 10^4, 2\times 10^4)$, $\zeta(-1, 0, 1)$

(e) $\Theta = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$, $\omega = 10$
$Q(-10^4, 0, 10^4)$, $\zeta(-1, 0, 1)$.



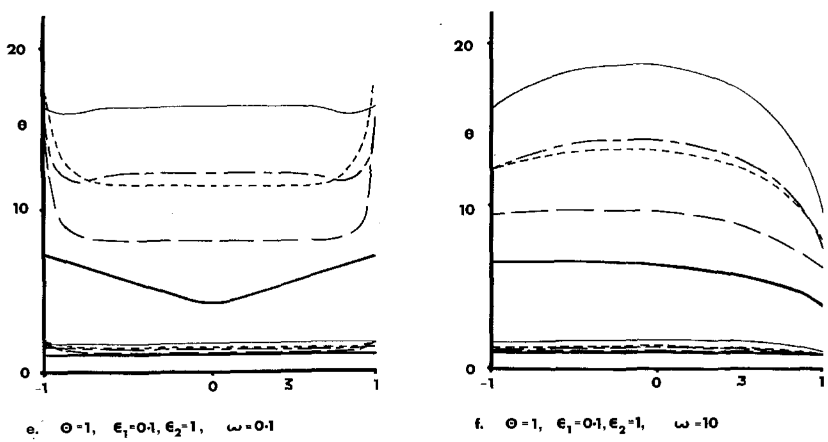


Figure 2. *Temperature distribution.*

Cases a, b, e, f: $N = 0.01$, upper 5 curves: $M = 0$
$N = 100$, lower 5 curves: $J = 1$; $J = 0.5$; $J = -0.5$; $J = 0$ } $M = 10$

Cases c, d: $N = 0.01$ as above
$N = 100$, $0 < M < 10$, all J …

(a) $\Theta = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\omega = 0.1$.
$\theta(0, 10, 20)$, $\zeta(-1, 0, 1)$.

(b) $\Theta = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\omega = 10$.
$\theta(0, 10, 20)$, $\zeta(-1, 0, 1)$.

(c) $\Theta = 10$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\omega = 0.1$.
$\theta(0, 10, 20)$, $\zeta(-1, 0, 1)$.

(d) $\Theta = 10$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\ = 10$.
$\theta(0, 10, 20)$, $\zeta(-1, 0, 1)$.

(e) $\Theta = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$, $\omega = 0.1$.
$\theta(0, 10, 20)$, $\zeta(-1, 0, 1)$.

(f) $\Theta = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$, $\omega = 10$.
$\theta(0, 10, 20)$, $\zeta(-1, 0, 1)$.

Analysis then shows that

$$Q = \frac{2(1-\Theta^4)}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2} - \frac{2M^2}{N}\,\frac{\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2}$$

$$\times\left\{P^2 - \frac{2P\sinh M}{M\cosh M-\sinh M} + \frac{M\sinh 2M}{2(M\cosh M-\sinh M)^2}\right\}$$

$$+\frac{M^2}{N}\left\{P^2\zeta - \frac{2P\sinh M\zeta}{M\cosh M-\sinh M} + \frac{M\sinh 2M\zeta}{2(M\cosh M-\sinh M)^2}\right\},$$

$$\theta^4 = \frac{\frac{1}{2}}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2}\left\{\left(3\omega+\frac{4}{\varepsilon_2}-2\right)+\left(3\omega+\frac{4}{\varepsilon_1}-2\right)\Theta^4+3\omega\zeta(\Theta^4-1)\right\}$$

$$-\frac{MP}{2N\omega(M\cosh M-\sinh M)}\{(M^2-3\omega^2)\cosh M\zeta+3\omega^2\cosh M\}$$

$$-\frac{M^2P^2}{4N\omega}\{\tfrac{3}{2}\omega^2(\zeta^2-1)-1\}$$

$$+\frac{M^2}{16N\omega(M\cosh M-\sinh M)^2}\{(4M^2-3\omega^2)\cosh 2M\zeta+3\omega^2\cosh 2M\}$$

$$+\frac{M^2}{2N}\left\{\frac{3\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)\omega\zeta+3\omega\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1\right)+2\left(\frac{2}{\varepsilon_2}-1\right)\left(\frac{2}{\varepsilon_1}-1\right)}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2}\right\}$$

$$\times\left\{P^2 - \frac{2P\sinh M}{M\cosh M-\sinh M} + \frac{M\sinh 2M}{2(M\cosh M-\sinh M)^2}\right\}.$$

The limiting case, $M\to 0$, corresponds to zero magnetic field and radiative Poiseuille flow in rdinary fluid mechanics. For this special case one finds

$$\frac{u}{U_0} = \tfrac{3}{2}(1-\zeta^2),$$

$$Q = \frac{2(1-\Theta^4)}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2} + \frac{3}{N}\left\{\zeta^3 - \frac{2\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2}\right\},$$

$$\theta^4 = \frac{\frac{1}{2}}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2}\left\{\left(3\omega+\frac{4}{\varepsilon_2}-2\right)+\left(3\omega+\frac{4}{\varepsilon_1}-2\right)\Theta^4-3\omega\zeta(\Theta^4-1)\right\}$$

$$+\frac{9}{4N\omega}\left\{\zeta^2-\frac{\omega^2}{4}(\zeta^4-1)\right\}$$

$$+\frac{3}{2N}\left\{\frac{3\left(\frac{1}{\varepsilon_2}-\frac{1}{\varepsilon_1}\right)\omega\zeta+3\omega\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}-1\right)+2\left(\frac{2}{\varepsilon_2}-1\right)\left(\frac{2}{\varepsilon_1}-1\right)}{3\omega+2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)-2}\right\}.$$

The general characteristics of the flows may be classified in the magnetohydrodynamic case according to the sign and magnitude of the parameter J.

(i) $J = 1$. In this case $E = 0$ and the electrodes must therefore be short circuited. Calculation of the Lorentz force, $\mathbf{j} \times \mathbf{B}$, shows that this always opposes the flow so that the flow pattern corresponds to that of an electromagnetic brake.

(ii) $0 < J < 1$. In this instance the mean current is positive and a finite resistance must therefore be inserted in the external circuit between the electrodes. Power may thus be extracted from the channel flow and the device corresponds to a magnetohydrodynamic generator.

(iii) $J = 0$. For this case $E = 1$ and the circuit is open. For given magnetic field $\mathbf{B}_0$ one may thus determine the mean fluid speed U_0 and the configuration has the characteristics of a flow meter.

(iv) $J < 0$. With negative values of the mean current an external power source must be placed between the electrodes and the characteristics are those of an electromagnetic pump.

The variations of velocity and induced magnetic field across the channel are completely unaffected by radiative effects, and thus as well-established results of Hartmann flow are not considered further here. The radiative flux and temperature distributions are presented in Figures 1 and 2 for several values of Hartmann number, radiative parameters ω and N, wall emissivities ε_1 and ε_2, temperature ratio Θ and each class of flow characterised by the sign and magnitude of J. In the absence of thermal conductivity the temperature slip at the walls should be noted. It is also observed that the Bouguer number has little effect upon the radiative flux whilst the magnitude of the Hartmann number and the ratio of the Reynolds number to Boltzmann number are of considerable significance. In the case of the temperature distribution the Bouguer number is also seen to be an important parameter.

Calculations have also been carried out for the same values of J, ω, N and M for the cases when (i) $\Theta = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$, (ii) $\Theta = 0.1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, and (iii) $\Theta = 10$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$. The results however portray similar effects to those illustrated in Figures 1 and 2.

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